Learn the Lagrangian: A Vector-Valued RKHS Approach to Identifying Lagrangian Systems

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***Abstract*—We study the modeling of Lagrangian systems with multiple degrees of freedom. Based on system dynamics, canonical parametric models require ad hoc derivations and sometimes simplification for a computable solution; on the other hand, due to the lack of prior knowledge in the system's structure, modern non-parametric models in machine learning face the curse of dimensionality, especially in learning large systems. In this paper, we bridge this gap by unifying the theories of Lagrangian systems and vector-valued reproducing kernel Hilbert space. We reformulate Lagrangian systems with kernels that embed the governing Euler-Lagrange equation—the Lagrangian kernels—and show that these kernels span a subspace capturing the Lagrangian's projection as inverse dynamics. By such property, our model uses only inputs and outputs as in machine learning and inherits the structured form as in system dynamics, thereby removing the need for the mundane derivations for new systems as well as the generalization problem in learning from scratches. In effect, it learns the system's Lagrangian, a simpler task than directly learning the dynamics. To demonstrate, we applied the proposed kernel to identify the robot inverse dynamics in simulations and experiments. Our results present a competitive novel approach to identifying Lagrangian systems, despite using only inputs and outputs.**

***Index Terms*— Lagrangian Systems, System Identification, Vector-Valued Reproducing Kernel Hilbert Space.**

I. Introduction

LAGRANGIAN systems constitute an important class of dynamical systems, covering various electro/mechanical applications. Euler-Lagrange equation, simplifying complicated dynamics, governs these systems by the variation of the Lagrangian with respect to the generalized coordinates. Canonically, Lagrangian systems are approximated with analytic models derived by analyzing the system's energy. However, though the analytic models are often linear in unknowns whose number is linearly proportional to the system's degrees of freedom (DOF), identifying general systems with analytic models suffers from computational complexity as the DOF increases (e.g. general robot dynamics with arbitrary links, elasticity, or no prior kinematic information), let alone the mundane derivations for each new system.

Machine learning, on the other hand, has proven as a successful and convenient tool in system identification, treating dynamical systems as blackbox and learning the mapping from inputs to outputs [1-5]. Despite such convenience, however, the necessary observations for generalization to unseen data impede learning general dynamical systems, in particular multi-input-multi-output systems. The curse of dimensionality occurs, because most approaches ignore the correlations between outputs. For a system with N outputs, the approximation error of learning N mappings independently can be, in the worst case, N times more than that of modeling a single vector-valued function. Modeling the correlation, or transfer learning, reduces the effective number of mappings to learn.

We are interested in the link between the *analytic models* and the *learning models*. In particular, we focus on the kernel methods [6-9], the algorithms based on reproducing kernel Hilbert space (RKHS), which have become ubiquitous for the elegancy and the ability to approximate arbitrary continuous functions [10]. Because of the duality between RKHS and feature space, these two approaches are equivalent in essence, yet with the nuance that the analytic models assume the description of the system's Lagrangian to be known. That is, the analytic models identify the Lagrangian, whereas the learning models identify the dynamics. Therefore, if the given Lagrangian formulation approximates well, the analytic models, inheriting the structure of Lagrangian systems the learning models lack, often enjoy better generalization. But still, the analytic models reach the bottleneck in computation and derivation when modeling large systems. This observation motivates us to design learning models that learn the Lagrangian, the truly indispensable unknown in a Lagrangian system.

The Lagrangian kernel is a general framework that unifies the theories of RKHS and Lagrangian systems. We reformulate Lagrangian systems in RKHS in which a bounded linear operator representing Euler-Lagrange equation exists, and study the geometric relationship between the Lagrangian and inverse dynamics (the mapping from generalized coordinates to generalized forces). We show if such a scalar RKHS $\mathcal{H}_L$ exists and contains the Lagrangian $\mathcal{L}$, then $\mathcal{H}_L$ includes the (scalar) inverse dynamics of each generalized coordinate as the projection

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of $\mathcal{L}$ on the subspace spanned by the scalar Lagrangian kernels in $\mathcal{H}_L$. Further, we show that these scalar Lagrangian kernels effectually induce a vector-valued Lagrangian kernel that spans a RKHS containing inverse dynamics as a vector-valued function. Finally, we indicate that the Lagrangian kernels model any Lagrangian system, because such RKHSs are dense in the function space of admissible Lagrangians. In short, our result provides a pathway to structure kernels for Lagrangian systems. In particular, the vector-valued Lagrangian kernels model the correlations between outputs, confining the learning complexity.

Modeling vector-valued functions has been treated mostly as multiple independent scalar problems. To further consider the outputs' correlations, some algorithms are modified to penalize the differences between outputs [11-14], controlling the size of the hypothesis space; other algorithms, e.g. locally weighted projection regression [15], can be intrinsically extended to vector-valued problems. Nevertheless, these methods rather induce biases or are *ad hoc*.

The theory of vector-valued RKHS, on the other hand, provides a natural extension. However, unlike scalar RKHSs, vector-valued RKHS has drawn little attention until last decade, though the mathematical studies dates back to earlier studies [16, 17]. Recently, because of the increasing need of multi-task learning and transfer learning, kernel methods based on scalar RKHS and the representer theorem [18] were further generalized [10, 19-22]. The designed vector-valued kernels, intrinsically considering the correlations between outputs, improve the performance [19, 23-26]. The characteristics of a kernel impact the convergence to the target function as the observations accumulate: The structure of the modelled function space, namely the respective norm in RKHS, controls the effective complexity of the hypothesis space, which decides the generalization. Therefore, by designing the parameterization of a kernel, the outcome can significantly improves, e.g. [27].

We demonstrate the Lagrangian kernels with the identification of robot inverse dynamics, which is widely used in model-based controls [28-30]. To investigate, we combine the proposed Lagrangian kernels with parametric basis for friction in a semi-parametric framework and study the differences between learning the Lagrangian and directly modeling the dynamics. Our results show, the Lagrangian kernels are more favorable, because the machine exploits Euler-Lagrange equation, thereby adopting a smaller hypothesis space.

This paper is organized as follows. In Section II, our main contribution, we reformulate Lagrangian systems in RKHS and design the Lagrangian kernels that embed the correlations and structure of inverse dynamics into the inner product of RKHS. Then we apply the Lagrangian kernels to identify robot dynamics in a semi-parametric framework to approximate systems that are not totally Lagrangian in Section III, and show the results of simulations and experiments in Section IV. Finally, we discuss the results and its potential applications in Section V, and give a short conclusion in Section VI.

II. Lagrangian Systems in Reproducing Kernel Hilbert Space and the Lagrangian Kernels

We reformulate Lagrangian systems in RKHS using the Lagrangian kernels. In the following, we first briefly review the essence of RKHS theory [10, 20] in Section II-A. Then in Section II-B, we show that the inverse dynamics of a particular generalized coordinate lies in the subspace spanned by the scalar Lagrangian kernels, and unify these relationships into the vector-valued Lagrangian kernel. Finally, we illustrate a family of Lagrangian kernels that learns arbitrary Lagrangian systems with only inputs and outputs in Section II-C.

A. *Reproducing* Kernel *Hilbert Space[10, 20]*

Let $\mathcal{X}$ be a metric space with probability measure μ_x and $\mathcal{Y}$ be a finite-dimensional real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$. $\mathcal{Y}^{\mathcal{X}}$ denotes the space of functions that maps from $\mathcal{X}$ to $\mathcal{Y}$, and $C(\mathcal{X};\mathcal{Y})$ denotes the Banach space of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ endowed with infinity norm. Given two normed linear space A and B, we denote by $L(A;B)$ the Banach space of bounded linear operators from A to B, by $L(A)$ the bounded linear operators from A to A, and by $L_+(A)$ the bounded positive semi-definite operators defined on A. For simplicity, we use the convention $\mathbb{N}_M := \{1,...,M\}$, where $M \in \mathbb{N}_+$ and $\mathbb{N}_+$ is the set of strictly positive numbers; a matrix $A \in \mathbb{R}^{N\times N}$ with (i, j) entry as A_{ij} is denoted as $A = (A_{ij})_{i,j\in\mathbb{N}_N}$.

A RKHS is related to a special class of functions, called kernels of positive type.

Definition 1

Given a metric space $\mathcal{X}$ and a Hilbert space $\mathcal{Y}$ as defined previously, a kernel $K : \mathcal{X}\times\mathcal{X} \to L(\mathcal{Y})$ is said to be of positive type, *if K is positive semidefinite, i.e.*

$$\sum_{i,j\in\mathbb{N}_M} c_i c_j \left\langle y_i, K(x_i, x_j) y_j \right\rangle_{\mathcal{Y}} \geq 0, \qquad (1)$$

for any $M \in \mathbb{N}_+$, $\{x_j \mid j \in \mathbb{N}_M\} \subseteq \mathcal{X}$, $\{y_j \mid j \in \mathbb{N}_M\} \subseteq \mathcal{Y}$, and $c_1,..,c_M \in \mathbb{R}$.

In particular, for $\mathcal{Y} = \mathbb{R}$, Definition 1 defines the conventional scalar kernels $k : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$, satisfying

$$\sum_{i,j\in\mathbb{N}_M} c_i c_j k(x_i, x_j) \geq 0. \qquad (2)$$

For a general vector-valued kernel K, (1) can also be treated as a scalar kernel $K_{nm}(x_i, x_j)$ defined on $\mathcal{X}\times\mathcal{Y}$ as

$$\sum_{i,j\in\mathbb{N}_M} \sum_{n,m\in\mathbb{N}_N} c_i c_j y_{i,n} y_{j,m} K_{nm}(x_i, x_j) \geq 0 \qquad (3)$$

in which $K_{nm}(x_i, x_j)$ is the (n,m) entry of the matrix $K(x_i, x_j)$ and N is the dimension of $\mathcal{Y}$. That is, the equivalent matrix $(K_{nm}(x_i, x_j))_{in,jm} \in L_+(\mathbb{R}^{NM})$ in (3) has the property of Gramian matrix, which induces RKHS $\mathcal{H}$ by inner product.

Definition 2

A Hilbert space $\mathcal{H}$ of functions $f \in \mathcal{Y}^{\mathcal{X}}$ endowed with the inner

product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ *and norm* $\|\cdot\|_{\mathcal{H}}$ *is called a* reproducing kernel Hilbert space *on* $\mathcal{X}$ *, if there is a map* $K_x : \mathcal{X} \to L(\mathcal{Y};\mathcal{H})$ *satisfying* $\forall x \in \mathcal{X}$

$$\langle f(x), y\rangle_{\mathcal{Y}} = \langle f, K_x y\rangle_{\mathcal{H}}, \ \forall f \in \mathcal{H} \ . \quad (4)$$

In particular, the reproducing kernel $K : \mathcal{X}\times\mathcal{X} \to L(\mathcal{Y})$ *is defined as*

$$K(x,u) = K_x^* K_u, \ \forall x,u \in \mathcal{X}, \quad (5)$$

and

$$f(x) = K_x^* f, \ \forall f \in \mathcal{H} \quad (6)$$

where $K_x^* : \mathcal{X} \to L(\mathcal{H};\mathcal{Y})$ *is the adjoint of* K_x *.*

We note that $K : \mathcal{X}\times\mathcal{X} \to L(\mathcal{Y})$ denotes the kernel function and $K_x : \mathcal{X} \to L(\mathcal{Y};\mathcal{H})$ denotes the feature map, and that $K(x,u)y$ is equal to $(K_u y)(x)$, where the second argument u is used as the index. Therefore, the Hilbert space $\mathcal{H}$ can be characterized as

$$\mathcal{H} = \overline{span}\{K_x y \mid x \in \mathcal{X}, y \in \mathcal{Y}\} \quad (7)$$

where the bar denotes the completion, with the following properties:

Proposition 1

1. A reproducing kernel is unique up to isometries.

2. A bounded reproducing kernel exists if and only if $\forall x \in \mathcal{X}$ *,* $f(x)$ *is a continuous linear map of* f *running through* $\mathcal{H}$ *.*

3. $K(x,u)$ *is of positive type* $\forall x,u \in \mathcal{X}$ *.*

4. $\mathcal{H} \subseteq C(\mathcal{X};\mathcal{Y})$ *if and only if* $K(x,u)$ *is locally bounded and* $\forall x \in X$ *,* K_x *is strongly continuous.*

The first and the third propositions are the properties of Hilbert space; the second is attributable to (6) and the last results from continuity, which is sufficient for modeling dynamics.

In addition, it can be easily shown that $K(x,u) = K(u,x)^*$ and $K(x,x) \in L_+(\mathcal{Y})$, i.e. $\|K(x,x)\|_{\mathcal{Y}}$ is a semi-norm of $\mathcal{H}$. In particular, we have $\forall x,u \in \mathcal{X}$, $y,v \in \mathcal{Y}$

$$\|K_x\|_{\mathcal{H}} = \|K(x,x)\|_{\mathcal{Y}}^{1/2} \quad (8)$$

and

$$\langle v, K(u,x)y\rangle_{\mathcal{Y}} = \langle K_u v, K_x y\rangle_{\mathcal{H}}; \quad (9)$$

By (9), the (n,m) element in the matrix $K(x,u)$,

$$K_{nm}(x,u) = \langle e_n, K(x,u)e_m\rangle_{\mathcal{Y}} = \langle K_x e_n, K_u e_m\rangle_{\mathcal{H}}, \quad (10)$$

is the inner product of the images in $\mathcal{H}$ of canonical basis $e_n, e_m \in \mathcal{Y}$. Therefore, the kernel $K(x,u)$ implicitly encodes the correlation between different coordinates in $\mathcal{Y}$. In short, (6) generalizes the reproducing property of scalar kernels $f(x) = \langle k_x, f\rangle_{\mathcal{H}}$ from the inner product to the locally bounded linear operator K_x^*.

Finally, some admissible constructions of RKHSs and the generalized representer theorem are reviewed below.

Proposition 2

Let G, K *be two reproducing kernels of RKHS as defined previously.*

1. The sum $K+G$ *is a reproducing kernel and the corresponding RKHS is the direct sum of the two RKHSs.*

2. The Hadamard product, $K \circ G$ *, is a reproducing kernel and the corresponding RKHS is the tensor product of the two RKHSs.*

3. Let $\mathcal{X}_0$ *be a compact Hausdorff space,* $\mathcal{Y} \in \mathbb{R}^N$ *,* $\Gamma_p : \mathcal{X} \to \mathcal{X}_0$ *for all* $p \in \mathbb{N}_N$ *. Given a scalar kernel* $G : \mathcal{X}_0 \times \mathcal{X}_0 \to \mathbb{R}$ *, then*

$$K(x,u) = \left(G(\Gamma_p x, \Gamma_q u)\right)_{p,q\in\mathbb{N}_N} \quad (11)$$

is a vector-valued reproducing kernel.

Proposition 3 (Representer Theorem)

Given a metric space $\mathcal{X}$ *, a Hilbert space* $\mathcal{Y}$ *, a reproducing kernel* $K : \mathcal{X}\times\mathcal{X} \to L(\mathcal{Y})$ *of (vector-valued) RKHS* $\mathcal{H}$ *, a strictly monotonically increasing real-valued function* g *on* $[0,\infty)$ *, an arbitrary cost function* $c : (\mathcal{X}\times\mathcal{Y}\times\mathcal{Y})^M \to \mathbb{R}\cup\{\infty\}$ *, and a set of samples* $\{(x_i, y_i) \mid x_i \in \mathcal{X}, y_i \in \mathcal{Y}\}_{i\in\mathbb{N}_M}$ *, the minimizer of*

$$\min_{f\in\mathcal{H}} c((x_1, y_1, f(x_1)), \ldots, (x_M, y_M, f(x_M))) + g(\|f\|_{\mathcal{H}}) \quad (12)$$

admits a representation of the form

$$f = \sum_{i\in\mathbb{N}_I} K_{x_i}\alpha_i \ . \quad (13)$$

where $\alpha_i \in \mathcal{Y}$ and $M > 0$.

B. Lagrangian Systems: A Reproducing Kernel Hilbert Space Formulation

Consider an *N*-DOF Lagrangian system. Let $q \in \mathbb{R}^N$ be the generalized coordinates and $\tau \in \mathbb{R}^N$ be the generalized forces. Define $\mathcal{X} = \{x = (q, \dot{q}, \ddot{q}) \mid \|x\|_\infty < \infty\}$ as the set of all the possible states with the probability measure μ_x, $\mathcal{Y} \subseteq \mathbb{R}^N$ as the space of generalized forces, and the subset $X = \{(q,\dot{q}) \mid q, \dot{q} \in \mathcal{X}\}$. For simplicity, with the abuse of notations, we write, for example, $q \in \mathcal{X}$, $x \in X$, or $x \in \mathcal{X}$.

The system is governed by the variation of the Lagrangian in Euler-Lagrange equation [31]:

$$\left(\frac{d}{dt}\frac{\partial}{\partial\dot{q}_n} - \frac{\partial}{\partial q_n}\right)\mathcal{L}(q,\dot{q},t) = \tau_n, \quad (14)$$

in which $\mathcal{L} = E_K - E_P$ is the Lagrangian with kinematic energy E_K and potential energy E_P, and $n \in \mathbb{N}_N$. For (14), the choice of the Lagrangian is not unique; for example,

$$\mathcal{L}'(q,\dot{q},t) = \mathcal{L}(q,\dot{q},t) + dF(q,t)/dt \quad (15)$$

is also a valid Lagrangian, in which $F(q,t)$ is a differentiable function.

To formulate (14) in RKHS, first, we exploit the reproducing property (4) to show that the differential operator (14) is equivalent to a bounded linear operator in RKHS, if the Lagrangian is implicit in time, i.e. $\mathcal{L}(q,\dot{q},t) = \mathcal{L}(q,\dot{q})$.

Proposition 4

Assume $\mathcal{L}(q,\dot{q},t)=\mathcal{L}(q,\dot{q})$. *There exists a scalar RKHS* $\mathcal{H}_L$ *endowed with map* $k_{L,x}: X \to \mathcal{H}_L$ *such that* $\mathcal{L}\in\mathcal{H}_L$ *satisfying*

$$\tau_n(q,\dot{q},\ddot{q}) = \left\langle \mathcal{L}, T_n k_{L,x} \right\rangle_{\mathcal{H}_L}, \tag{16}$$

in which $T_n \in L(\mathcal{H}_L)$ *is defined as*

$$\left\langle f, T_n k_{L,x} \right\rangle_{\mathcal{H}_L} := \left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n}\right) f(q,\dot{q}) \tag{17}$$

for all $f\in\mathcal{H}_L$.

Proof: Shown in Appendix A. □

Next, by Proposition 4**,** we show $T_n k_{L,x}$ spans a subspace in $\mathcal{H}_L$ on which τ_n is the projection of $\mathcal{L}$.

Proposition 5

Define

$$k_{Tn,x} := T_n k_{L,x}, \tag{18}$$

$$\mathcal{H}_{T,n} := \overline{span}\{k_{Tn,x} \mid x\in\mathcal{X}\} \tag{19}$$

and

$$\mathcal{H}_T := \bigoplus_{n\in\mathbb{N}_N} \mathcal{H}_{T,n}. \tag{20}$$

Under the assumption in Proposition 4,

1. *The admissible Lagrangians* $\mathcal{L}$ *of (14) form an affine space in* $\mathcal{H}_L$*, i.e.* $\mathcal{L}=\mathcal{L}_p+\mathcal{L}_h$*, where* $\mathcal{L}_p\in\mathcal{H}_T$ *and* $\mathcal{L}_h\in\mathcal{H}_L\setminus\mathcal{H}_T$.
2. $\mathcal{H}_{T,n}\subseteq\mathcal{H}_L$ *is a RKHS with kernel* $k_{Tn,x}:\mathcal{X}\to\mathcal{H}_{T,n}$ *such that unique* $\tau_n\in\mathcal{H}_{T,n}$ *exists satisfying*

$$\tau_n(x) = \left\langle \tau_n, k_{Tn,x} \right\rangle_{\mathcal{H}_{T,n}}, \forall x\in\mathcal{X}; \tag{21}$$

3. $\tau_n\in\mathcal{H}_{T,n}$ *is the projection of* $\mathcal{L}$ *on the* $\mathcal{H}_{T,n}$.

Proof: Shown in Appendix A. □

In the following, we use $T_n k_{L,x}$ when emphasizing $\mathcal{H}_L$ and $k_{Tn,x}$ when emphasizing $\mathcal{H}_{T,n}$.

Using the above results, we show our main result: the scalar RKHSs $\mathcal{H}_{T,n}$ induces a vector-valued RKHS $\mathcal{H}_{vT}$ that contains the inverse dynamics $\tau=(\tau_n)_{n\in\mathbb{N}_N}$ as a vector-valued function.

Theorem 1

Let $\mathcal{H}_L$ *be defined previously, and define*

$$\mathrm{K}_T(x,u) := \left(\left\langle T_m k_{L,u}, T_n k_{L,x} \right\rangle_{\mathcal{H}_L}\right)_{n,m\in\mathbb{N}_N} \tag{22}$$

The reproducing kernel $\mathrm{K}_T(x,u):\mathcal{X}\times\mathcal{X}\to L(\mathbb{R}^N)$ *defines a vector-valued RKHS* $\mathcal{H}_{vT}$ *such that there exists* $\tau\in\mathcal{H}_{vT}$ *satisfying* $\tau(x)=K^*_{T,x}\tau$ *for all* $x\in\mathcal{X}$. *In particular, for* $x,u\in\mathcal{X}$

$$\begin{aligned}\mathrm{K}_T(x,u) = {} & \left(\frac{\partial x}{\partial q}\right)^T \nabla_{xu} k_L \left(\frac{\partial u}{\partial q}\right) - \left(\frac{\partial x}{\partial \dot{q}}\right)^T \left(\frac{d}{dt_x}\nabla_{xu}k_L\right)\left(\frac{\partial u}{\partial q}\right) \\ & -\left(\frac{\partial x}{\partial q}\right)^T \left(\frac{d}{dt_u}\nabla_{xu}k_L\right)\frac{\partial u}{\partial \dot{q}} + \left(\frac{\partial x}{\partial \dot{q}}\right)^T \left(\frac{d^2}{dt_x dt_u}\nabla_{xu}k_L\right)\frac{\partial u}{\partial \dot{q}}\end{aligned} \tag{23}$$

in which $\nabla_{xu}k_L := (\partial^2 k_L / \partial x_n \partial u_m)_{n,m\in\mathbb{N}_N}$ *and* d/dt_x *is the time derivative with respect to the variable* x.

Proof: By Proposition 2, the construction of K_T in (22) is a valid vector-valued reproducing kernel. To prove $\tau\in\mathcal{H}_{vT}$, by representer theorem, let

$$\mathcal{L}_p = \sum_{n\in\mathbb{N}_N} \int_{\mathcal{X}} \alpha_n(u) T_n k_{L,u} du \tag{24}$$

for some α_n, and rewrite (16) as

$$\tau_n(x) = \left\langle \mathcal{L}_p, T_n k_{L,x} \right\rangle_{\mathcal{H}_L} = \sum_{m\in\mathbb{N}_N} \int_{\mathcal{X}} \alpha_n(u) \mathrm{K}_{T,nm}(x,u) du. \tag{25}$$

Then, from Definition 2, (25) can be collected in vector form as

$$\tau(x) = K^*_{T,x}\left(\int_{\mathcal{X}} K_{T,u}\alpha(u)du\right) \tag{26}$$

in which $K^*_{T,x}:\mathcal{H}_{vT}\to\mathbb{R}^N$ is the evaluation map and $\alpha(u) := (\alpha_n(u))_{n\in\mathbb{N}_N}$. Therefore, defining

$$\tau := \int_{\mathcal{X}} K_{T,u}\alpha(u)du, \tag{27}$$

we have $\tau\in\mathcal{H}_{vT}$ satisfying $\tau(x)=K^*_{T,x}\tau$ for all $x\in\mathcal{X}$.

To derive the exact formulation of $\mathrm{K}_{vT}(x,u)$ from $k_L(x,u)$, we first see from definition in (17),

$$\left\langle T_m k_{L,u}, T_n k_{L,x} \right\rangle_{\mathcal{H}_L} = \left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n}\right)_x \left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_m} - \frac{\partial}{\partial q_m}\right)_u k_L(x,u) \tag{28}$$

Second, since $x=(q,\dot{q},\ddot{q})$ fulfilling $d(\partial_{\dot{q}_n} x)=0$, we have

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n}\right)_x k_L(x,u) = \left(\frac{d}{dt_x}\frac{\partial k_L}{\partial x}\right)\frac{\partial x}{\partial \dot{q}_n} - \frac{\partial k_L}{\partial x}\frac{\partial x}{\partial q_n}. \tag{29}$$

Finally, combining (28) and (29), and using chain rule of deviation, we arrive at the equation (23). □

We call (18), or its equivalence

$$k_{Tn}(x,u) = \left\langle T_n k_{L,u}, T_n k_{L,x} \right\rangle_{\mathcal{H}_L} \tag{30}$$

the *scalar Lagrangian kernel* of the *n*th generalized coordinate associated with $\mathcal{H}_L$ and (22) the *vector-valued Lagrangian kernel* associated with $\mathcal{H}_L$. The Lagrangian kernels, induced from $\mathcal{H}_L$ satisfying (16), provide the reproducing properties $\tau_n(x)=\left\langle \tau_n, k_{Tn,x}\right\rangle_{\mathcal{H}_{T,n}}$ and $\tau(x)=K^*_{T,x}\tau$. As such, the Lagrangian kernels reformulate an *N*-DOF Lagrangian system into the following representation:

$$\mathcal{L}(q,\dot{q}) = \mathcal{L}_p(q,\dot{q}) = \sum_{n\in\mathbb{N}_N} \int_{\mathcal{X}} \alpha_n(u) T_n k_{L,u}(x) du, \tag{31}$$

$$\tau(q,\dot{q},\ddot{q}) = \int_{\mathcal{X}} K_{T,u}(x)\alpha(u)du \tag{32}$$

for some $\alpha_n:\mathcal{X}\to\mathbb{R}$ and $\alpha=(\alpha_n)_{n\in\mathbb{N}_N}$, or

$$\tau_n(q,\dot{q},\ddot{q}) = \int_{\mathcal{X}} k_{Tn,u}(x)\beta(u)du \tag{33}$$

for some $\beta_n:\mathcal{X}\to\mathbb{R}$, in which by (17)

$$T_n k_{L,u}(x) := \left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n}\right)_u k_L(u,x), \tag{34}$$

$K_{T,u}(x)$,and $k_{Tn,u}(x)$ are defined in (30), and (22), respectively.

In particular, we note that $T_n k_{L,u}(x) = \langle T_n k_{L,u}, k_{L,x} \rangle_{\mathcal{H}_L}$ is not a kernel, as opposed to $k_{Tn,u}(x) = \langle T_n k_{L,u}, T_n k_{L,x} \rangle_{\mathcal{H}_L}$ which is valid kernel. The representations (31)–(33) converge pointwisely to the true functions, in which the summation over N is due to (20); though the coefficients α_n, β_n may not be unique, (31)–(33) are unique. Finally, we note that k_L is defined on X, whereas the Lagrangian kernels k_{Tn} and K_T are defined on $\mathcal{X}$, including acceleration $\ddot{q}$. This results from being the RKHS for τ_n, because dynamics involves position, velocity, and acceleration of the generalized coordinates.

Based on the reproducing property, a set of scalar Lagrangian kernels $\{k_{Tn} \mid n \in \mathbb{N}_N\}$ associated with a common $\mathcal{H}_L$ induces a vector-valued Lagrangian kernel K_T which models the correlations between different joints as indicated in the (m, n) entry of (22). Therefore the matrix function $\mathrm{K}_T(x,u)$ is diagonal if and only if $\mathcal{H}_{T,n} \perp \mathcal{H}_{T,m}$ for $n \neq m$, which depends on the choice of $\mathcal{H}_L$. Similarly, as long as only dynamics are concerned, we can generalize this concept to a set of scalar kernels with $\mathcal{H}_{L_n}$, or its equivalent vector-valued kernel, as

$$\tilde{K}_T(x,u)_{nm} = \delta_{nm} \langle T_n k_{L_n,u}, T_n k_{L_n,x} \rangle_{\mathcal{H}_{L_n}}, \qquad (35)$$

in which different RKHSs are used to derive the scalar Lagrangian kernel for each coordinate and δ_{nm} is Kronecker delta. The scalar Lagrangian kernels (30) and (35) preserve the structure due to Euler-Lagrange equation, and the vector-valued Lagrangian kernels (22) further encode the correlations between different coordinates.

C. Interpretation of Lagrangian Kernel

The RKHS formulation of Euler-Lagrange equation provides geometric insights into the relationship between the Lagrangian and inverse dynamics, as summarized in Fig. 1. Firstly, given a RKHS $\mathcal{H}_L$ in which the differential operator T_n in Euler-Lagrange equation can be defined (i.e. the kernel k_L of $\mathcal{H}_L$ is at least second-order differentiable), the admissible Lagrangians $\mathcal{L}$ form an affine space $\mathcal{L}_p + \mathcal{L}_h$ whose projection on $\mathcal{H}_{T,n}$, the span of the Lagrangian kernel $k_{Tn,u}$, defines the inverse dynamics τ_n. In addition, it identifies the solution of system identification. Because other solutions with nonzero $\mathcal{L}_h$ cannot be discriminated insofar as only dynamics are measured, only the minimum norm solution $\mathcal{L}_p$ is tractable. Finally, other functions in $\mathcal{H}_L$ satisfy (16) (e.g., a trivial choice $\mathcal{L}_p + \mathcal{L}_h$), but the choice of $\tau_n \in \mathcal{H}_{T,n}$ inherits nice properties from the representer theorem, providing a unified view of the Lagrangian and inverse dynamics.

A simple illustration of the Lagrangian kernels is the robot dynamics model based on known kinematics parameters. If the kinematic parameters of a robot are completely known, the

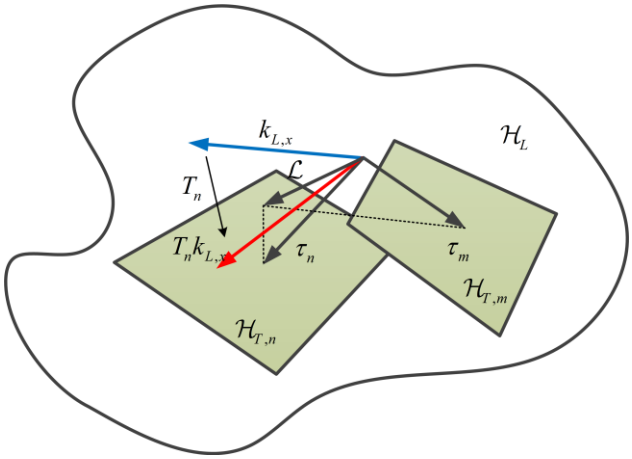

Fig. 1. The relationship between $\mathcal{H}_L$ and $\mathcal{H}_{T,n}$ for an N-DOF Lagrangian system with the Lagrangian $\mathcal{L}$. For a scalar RKHS $\mathcal{H}_L$ with kernel $k_{L,x}$ such that $\mathcal{L} \in \mathcal{H}_{\mathcal{L}}$, the inverse dynamics of the nth generalized coordinate $\tau_n \in \mathcal{H}_{T,n}$ is the projection of $\mathcal{L}$ on $\mathcal{H}_{T,n}$, in which $\mathcal{H}_{T,n} \subseteq \mathcal{H}_L$ is spanned by $T_n k_{L,x}$, $T_n : \mathcal{H}_L \to \mathcal{H}_L$ is the bounded linear operator due to the Euler-Lagrange equation, and $n, m \in \mathbb{N}_N$.

Lagrangian is linear in the inertial parameters [32] admitting the form $\mathcal{L}(q,\dot{q}) = \langle \theta, \varphi_x \rangle$,where $\varphi_x \in \mathbb{R}^{10N}$ is the function of the kinematics and states, and $\theta \in \mathbb{R}^{10N}$ is a constant vector of the inertial parameters to be identified. That is, in our new RKHS formulation, $\mathcal{H}_L = \mathbb{R}^{10N}$ and $k_{L,x} = \varphi_x$, and the scalar Lagrangian kernels $k_{Tn,x} = T_n \varphi_x$. Let $\Omega(x) := [k_{T_1,x} \ldots k_{T_N,x}]^T$. The induced vector-valued Lagrangian kernel is then $K_T(x,u) = \Omega(x)\Omega(u)^T$. Using the representation, the inverse dynamics can be written linearly as $\tau(x) = \Omega(x)\theta$, identical to the conventional result [33] derived from the Newton-Euler method.

D. Universal Lagrangian Kernels

The requirement that k_L is second-order differentiable is not restrictive, as universal analytic kernels exist, uniformly approximating any continuous functions on compact support [10]. To approximate arbitrary Lagrangian functions, we consider a family of radial basis function (rbf) kernels with different feature maps as candidates for k_L.

Corollary 1

Let $\phi : \mathbb{R}^N \to \mathbb{R}^{d_1}$ *and* $\psi : \mathbb{R}^N \to \mathbb{R}^{d_2}$ *be diffeomorphisms such that* $d(\partial\psi)/dt = 0$, *and define* $\tilde{x} := (\phi_x, \psi_x)$ *with* $\phi_x = \phi(q_x)$ *and* $\psi_x = \psi(\dot{q}_x)$ *for* $x \in X$. *Let*

$$k_L(x,u) = \exp\left(-\frac{\|\phi_x - \phi_u\|^2 + \|\psi_x - \psi_u\|^2}{2}\right) \qquad (36)$$

be the reproducing kernel of $\mathcal{H}_L$ *for* $x, u \in X$. *Then the associated vector-valued reproducing kernel* K_T *of* $\mathcal{H}_{vT}$ *is*

$$\mathrm{K}_T(x,u) = k_L(\mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_3 + \mathrm{K}_4), \qquad (37)$$

in which

$$K_1 = \partial\phi_x^T[I-(\phi_x-\phi_u)(\phi_x-\phi_u)^T]\partial\phi_u$$
$$K_2 = -\partial\psi_x^T[c_x(\psi_x-\psi_u)(\phi_x-\phi_u)^T-(\dot\psi_x(\phi_x-\phi_u)^T+(\psi_x-\psi_u)\dot\phi_x^T)]\partial\psi_u$$
$$K_3 = \partial\phi_x^T[c_u(\phi_x-\phi_u)(\psi_x-\psi_u)^T-(\dot\phi_u(\psi_x-\psi_u)^T+(\phi_x-\phi_u)\dot\psi_u^T)]\partial\psi_u$$
$$K_4 = \partial\psi_x^T\{c_{xu}[(\psi_x-\psi_u)(\psi_x-\psi_u)^T-I]+(\dot\psi_u\dot\psi_x^T+\dot\psi_x\dot\psi_u^T)$$
$$-c_x[\dot\psi_u(\psi_x-\psi_u)^T+(\psi_x-\psi_u)\dot\psi_u^T]-c_y[\dot\psi_x(\psi_x-\psi_u)^T+(\psi_x-\psi_u)\dot\psi_x^T]\}\partial\psi_u$$

$c_x=(\tilde x-\tilde u)^T\dot{\tilde x}$, $c_y=(\tilde x-\tilde u)^T\dot{\tilde u}$, $c_{xu}=(c_xc_y-\dot{\tilde x}^T\dot{\tilde u})$, $I\in\mathbb{R}^{N\times N}$ *is the identity matrix, and* ∂ *denotes partial derivative* $\partial/\partial x$.

The proof is only technical by applying Theorem 1, so we omit it here.

Despite the requirement $d(\partial\psi)/dt=0$, (36) includes a wide range of kernels useful in modeling dynamics. For example, choosing

$$\begin{cases}\phi(q)=\sigma^{-1/2}q\\ \psi(\dot q)=\sigma^{-1/2}c_q\dot q\end{cases} \tag{38}$$

(36) gives the standard rbf kernel with scaling factor $\sigma>0$ and normalization constant c_q. Or for modeling robots with rotary joints, an alternative could be

$$\begin{cases}\phi(q)=\sigma^{-1/2}(\cos(q),\sin(q))\\ \psi(\dot q)=\sigma^{-1/2}c_q\dot q\end{cases} \tag{39}$$

which considers the trigonometric functions.

III. Application to Identifying Robot Inverse Dynamics

A. A Semi-parametric Framework

As an application of the Lagrangian kernels, we identify the inverse dynamics for general holonomic rigid-body robots with frictions, and link it to the identification of general Lagrangian systems. Let the generalized force in the dynamics equation (14) be separated into

$$\tau=\tau_a+\tau_f \tag{40},$$

where $\tau_a\in\mathbb{R}^N$ denotes the actuator force and $\tau_f\in\mathbb{R}^N$ denotes friction and unmodelled dynamics. The inverse model useful in control is the mapping $\Gamma:\mathcal{X}\to\mathbb{R}^N$ which approximates τ_a from the states of the robot. Under rigid-body assumption, $\tau(q,\dot q,\ddot q)$ is well defined, but τ_f may not be a proper function on $\mathcal{X}$. Such setting is similar to the Lagrangian systems in which external forces enter.

Given M observations,

$$Z=\{(x_i,y_i)\,|\,x_i\in\mathcal{X},y_i\in\mathcal{Y},i\in\mathbb{N}_M\} \tag{41}$$

we solve the regularized least-square approximation in [34],

$$\min_{f\in\mathcal{H}_{dyn}\oplus\mathcal{H}_{fri}}\frac{1}{M}\sum_{i\in\mathbb{N}_M}\|f(x_i)-y_i\|_{\mathcal{Y}}^2+\gamma((1-\rho)\|f\|_{\mathcal{H}_{dyn}}^2+\rho\|f\|_{\mathcal{H}_{fri}}^2)$$

i.e.

$$\min_{f\in\mathcal{H}_{dyn}\oplus\mathcal{H}_{fri}}\frac{1}{M}\sum_{i\in\mathbb{N}_M}\|f(x_i)-y_i\|_{\mathcal{Y}}^2+\gamma\|f\|_{\mathcal{H}_{dyn}\oplus\mathcal{H}_{fri}}^2 \tag{42}$$

in which $\mathcal{H}_{dyn}$ with K_{dyn} is the RKHS to model Lagrangian systems τ, $\mathcal{H}_{fri}$ with K_{fri} is the RKHSs to model τ_f, and $\gamma>0$ is the regularized parameter. The scheme (42) uses the effective kernel of $\mathcal{H}_{dyn}\oplus\mathcal{H}_{fri}$,

$$K_{dyn\oplus fri}=(1-\rho)^{-1}K_{dyn}+\rho^{-1}K_{fri} \tag{43}$$

and $0<\rho<<1$ to approximate the semi-parametric problem

$$\min_{f\in\mathcal{H}_{dyn},b\in\mathbb{R}^B}\frac{1}{M}\sum_{i\in\mathbb{N}_M}\|f(x_i)+\Phi(x_i)b-y_i\|_{\mathcal{Y}}^2+\gamma\|f\|_{\mathcal{H}_{dyn}}^2 \tag{44}$$

with the solution

$$f_{\mathbf{z}}(x)=(1-\rho)^{-1}\sum_{i\in\mathbb{N}_M}K_{dyn}(x,x_i)a_i+\Phi(x_i)b\,, \tag{45}$$

in which

$$b=\rho^{-1}\sum_{i\in\mathbb{N}_M}\Phi(x_i)^Ta_i\,, \tag{46}$$

$\Phi\in\mathbb{R}^{N\times B}$ is the basis for modelling friction satisfying

$$K_{fri}(x,u)=\Phi(x)\Phi(u)^{\mathrm{T}}, \tag{47}$$

$B=\dim\mathcal{H}_{fri}<<\dim\mathcal{H}_{dyn}$ and $B<\infty$, and a_i are the unknowns to be solved in (42). The relaxation in (43) approximates the solution with $\rho=0$, so off-the-shelf kernel methods can be directly used, yet still give the semi-parametric solution (45).We chose (42) for simplicity, and the least-square regularization in (42) can be replaced with any cost function satisfying (12) (e.g. support vector regression).

We formulate the problem in vector-valued kernels, which also includes scalar kernels, because a set of scalar kernels $\{k_n(x,u)\,|\,n\in\mathbb{N}_N\}$ for N outputs equals to the vector-valued kernel $(\delta_{nm}k_n(x,u))_{n,m\in\mathbb{N}_N}$ as in (35). In addition, the framework (42) is a discriminative model based on the approximation theory, whereas a similar generative model based on the Bayesian theorem was presented in [35]. Therefore, the parametric rigid-body models [32, 33] can also be incorporated as Φ in (44).

This framework can also be used for the holonomic robots with structures, e.g. closed-loop robot, without referring to the exact independent coordinates. By virtue of virtual work and splitting the robots into tree subsystems [36] whose generalized coordinates are accessible, the dynamics can be written as

$$\tau=\int_{\mathcal{X}}[G^TK_T(x,u)G]\alpha(u)du+\Psi(x)\omega\,, \tag{48}$$

where $G=\partial q_{tr}/q_a$ is the Jacobian matrix due to kinematic constraints, $\Psi(x)\omega$ is the parametric model, e.g. the spatial force due to the platform or loads, q_a is the active coordinates, and q_{tr} is the accessible generalized coordinates of the subsystems. As a result, G^TK_TG form the new vector-valued Lagrangian kernel, with which the inverse dynamics problem can be solved in (42); the constrained dynamics of general Lagrangian systems yield a similar form [37].

B. Convergence

The learning with the proposed Lagrangian kernels in (42) is consistent, because it follows the framework of learning in

RKHS. Without specifying k_L, we qualitatively present the convergence of (42).

Theorem 2

Let $y = \tau_a = \tau - \tau_f$. Assume $\|\tau_a\|_y, \|\tau_f\|_y < \infty$, $\mathcal{H}_L$ is chosen so that $\mathcal{L} \in \mathcal{H}_L$, $\|\mathcal{L}_p\|_{\mathcal{H}_L} < \infty$, and T_n is bounded. Let $\mathcal{H}_{dyn} = \mathcal{H}_L$ be the RKHS with the Lagrangian kernels. There exists $\gamma^* > 0$ such that the estimator $f_{\mathbf{z}}$ in (45) by solving (42) with $0 < \gamma < \gamma^*$ converges to a solution

$$f_{\mu,\gamma} := \arg \min_{f \in \mathcal{H}_{dyn} \oplus \mathcal{H}_{fri}} \int_{\mathcal{X}} \| f(x) - y \|_y^2 \, d\mu_x + \gamma \| f \|^2_{\mathcal{H}_{dyn} \oplus \mathcal{H}_{fri}} \quad (49)$$

satisfying

$$\int_{\mathcal{X}} \| f_{\mu,\gamma}(x) - y \|_y^2 \, d\mu_x \le \min_{f \in \mathcal{H}_{fri}} \int_{\mathcal{X}} \| f(x) + \tau_f \|_y^2 \, d\mu_x \quad (50)$$

as $M \to \infty$ with probability 1.

Proof: Shown in Appendix B. □

Theorem 2 shows that the quality of the semi-parametric framework (42) in learning the dynamics depends on the quality of function space used to model friction and unmodeled dynamics. Specifically, with $0 < \gamma < \gamma^*$, the algorithm is consistent and converges to the solution that is at least as good as how $\mathcal{H}_f$ models the unmodeled terms.

IV. Simulations and Experiments of Robot Inverse Dynamics Identification

To demonstrate, we apply the Lagrangian kernels derived from rbf kernels in (36) with the least-square regularization in (42) to identify robot inverse dynamics in simulations and experiments. For friction, we adopt

$$K_{fri}(x,u)_{nm} = \delta_{nm}(\dot{q}_{x,n}\dot{q}_{u,n} + \tanh(\frac{\dot{q}_{x,n}}{\sigma_{f,n}})\tanh(\frac{\dot{q}_{u,n}}{\sigma_{f,n}}) + 1), \quad (51)$$

which is diagonal and corresponds to a block-diagonal basis $\Phi(x) \in \mathbb{R}^{N \times 3N}$, to model the coulomb and the viscous frictions.

In order to compare the proposed Lagrangian kernels and the state-of-the-art kernels, we test the identification using *only inputs and outputs*. For traditional kernels, the rbf kernel

$$k_{rbf}(x,u) := \exp(-\frac{\|x-u\|^2}{2\sigma}), \; x,u \in \mathcal{X} \quad (52)$$

and the rbf kernel with the trigonometric map for rotary joints

$$k_{rbfs}(x,u) := \exp(-\frac{\|\psi(q_x)-\psi(q_u)\|^2 + \|\dot{q}_x - \dot{q}_u\|^2 + \|\ddot{q}_x - \ddot{q}_u\|}{2\sigma}) \quad (53)$$

are denoted by *rbf* and *rbfs*, respectively, in which $\psi : \mathbb{R}^N \to \mathbb{R}^{2N}, \psi(q) = (\cos q, \sin q)$. For the Lagrangian kernels, we denote them by the prefix based on the construction (*s*-(35) or *v*-(22)) and the postfix based on the choice of k_L (*-rbf* (38) or *-rbfs*(39)) as *srbf, vrbf, srbfs*,and *vrbfs*. We omit (30) because it is a particular case of (35). The kernels *rbf* and *rbfs* are chosen to resemble kernels *srbf* and *vrbf, srbfs* and *svrbfs*, respectively, which use the same form of rbf kernels to model the Lagrangian. In addition, for benchmark, we implement the finite-dimensional kernel *pol* in [34] and the simple independent joint model *motor* in the form

$$c_1\ddot{q}_n + c_2\dot{q}_n \quad (54)$$

for joint *n*, where c_1 and c_2 are the unknowns to be identified. We omit the general Euler-Lagrange model, because it could be exponentially complex for general robots when without kinematic knowledge.

In the semi-parametric framework with the kernel *fri* defined in (51), the symbol + is used to combine two kernels in the form of (43), in which the left and the right arguments are $\mathcal{H}_{dyn}$ and $\mathcal{H}_{fri}$, respectively. For example, *rbf* stands for using (52) alone, and *rbf+fri* denotes the semi-parametric scheme.

In implementation, we take $\|\dot{q}\|_\infty \le 1$ $\|\ddot{q}\|_\infty \le 1$. For *rbf* and *rbfs*, the normalization is pre-computed before evaluating (52) and (53), whereas, for Lagrangian kernels, the normalization is implemented as $c_{\dot{q}}$ in (38) and (39).

A. Simulations

We show prediction error with respect to the complexity of the underlying model (i.e. the DOF) in two scenarios: with or without measurement noise and friction. For each DOF, 100 different serial rotary robots with random kinematic and inertial parameters, gear ratios, and friction magnitude are used as reference models, in which the generalized coordinates are the joints of the robot, and the parameters are sampled according to a bounded uniform probability distribution such that all the parameters are physically feasible. For example, the inertia matrix is always positive definite.

For each robot, $M = 500$ training data and $M_{val} = 15000$ validating data, with the angular positions, velocities, and accelerations uniformly sampled from a bounded set, were generated using Newton-Euler method; the torque τ was normalized to satisfy $\|\tau\|_\infty \le 1$ for comparison. The adopted noise was additive zero-mean Gaussian with standard variation 0.05 in the same unit as the normalized torque; the viscous and the Coulomb friction were independently modeled for each joint as the force linear to the joint velocity and the sign function, respectively.

To evaluate performance, the prediction error over all of the generalized coordinates was defined by root-mean-square (RMS), $N^{-1/2}\|y_i - f_{\mathbf{z}}(x_i)\|_y$, over M_{val} validation samples. For the least-square semi-parametric framework (42), the optimal parameters σ, $\sigma_{f,n}$, and γ were chosen by 5-fold cross-validation, if not particularly specified; ρ was fixed as $10^{-3} trace(\mathbf{K}_{dyn}) / trace(\mathbf{K}_{fri})$, where $\mathbf{K}_{dyn}, \mathbf{K}_{fri} \in \mathbb{R}^{NM \times NM}$ are the empirical kernel matrices. We remark that for the scalar kernels (*rbf, rbfs, svrbf* and *svrbfs*) *N* independent scalar regression problems are solved.

Fig.2 shows the RMS errors of prediction of ideal robot dynamics without friction and noises, in which $\gamma = 10^{-12}$ and the parameter σ is determined by cross-validation. The kernels

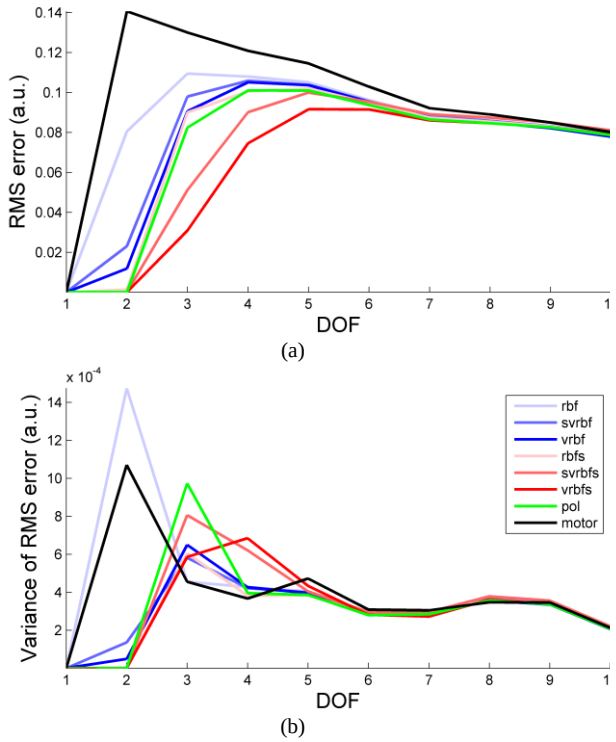

Fig. 2. RMS error of prediction in learning inverse dynamics of ideal rigid-body robot. (a) mean (b) variance. The lines are the linear interpolations of the simulation results.

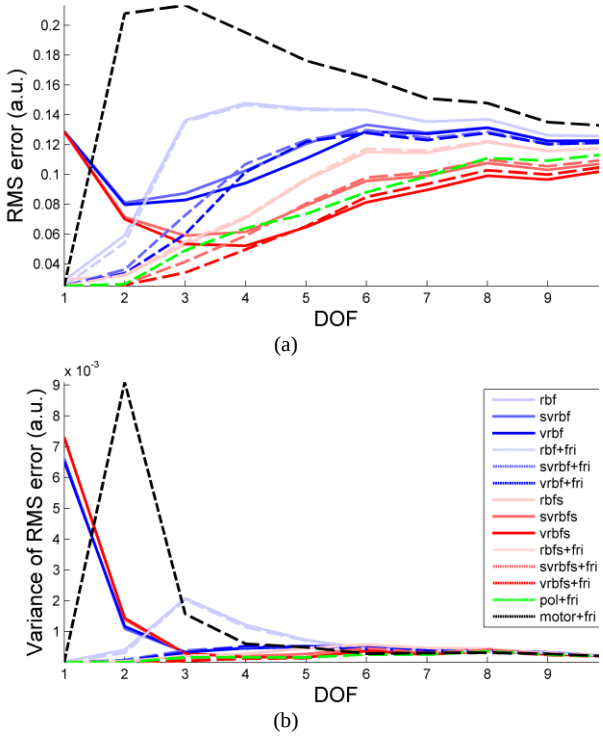

Fig. 3. RMS error of torque prediction in learning ideal robot inverse dynamics with simulated friction and Gaussian measurement noise. (a) mean (b) variance. The lines are the linear interpolations of the simulation results.

with the trigonometric map (*rbfs, svrbfs, vrbfs)* outperform that without one (e.g. *rbf, svrbf, svrbfs*), because the characteristics of rotary joints are better captured. In addition, the performance increases as the structure of Lagrangian systems are better modeled (e.g. from *rbf, srbf,* to *vrbf*). Compared with *rbf* and *rbfs*, the Lagrangian kernels consistently perform better, among which the vector-valued kernels *vrbf* and *vrbfs* are superior due to the modeled correlations.

Fig.3 shows the RMS errors of prediction of ideal robot dynamic s with friction and noises, in which the Coulomb and viscous friction are modeled as described with random size. Compared with Fig.2, the Lagrangian kernels (*svrbf, vrbf, svrbfs,* and *vrbfs*) perform arbitrarily worse when friction is large as with robots with small DOF, because these kernels only model Lagrangian systems. However, by incorporating friction model *fri*, these hybrid kernels (*svrbf+fri, vrbf+fri, svrbfs+fri,* and *vrbfs+fri*) perform as without the presence of friction in Fig.2. In addition, as the DOF increases, *fri* becomes less necessary. Because the size of friction is independent of DOF in the simulations, for robots with large DOFs, friction becomes relatively smaller in τ_a and therefore the learning of Lagrangian part dominates the performance.

B. *Experiments*

The models were empirically verified with the NTU robot arm (NTU Robotics Laboratory, National Taiwan University). Shown in Fig. 4, the NTU robot arm is a 6-DOF robotic manipulator driven by DC-micromotors with a high gear ratio. With current sensor, the robot arm is fed back by 10-kHz inner

Fig. 4. The 6-DOF NTU robot arm.

torque PI-controllers and outer 250-Hz position PD-controllers, and can be fed forward with additional torque commands. The joint position and current measurements were sampled at 500 Hz. To compute $\dot{q}$ and $\ddot{q}$, the sampled trajectories were filtered with a 3rd-order Butterworth filter with bandwidth 25 Hz and then differentiated. The joint torque was then obtained through the current measurement. To synchronize the measurements of the states and the joint torque, the joint torque was filtered by the same 3rd-order Butterworth filter.

To evaluate the performance, we used 10 trajectories (interpolated by a 5th-order polynomial spline) of different speeds, which randomly and smoothly traverses the whole work space for approximately 30–40 seconds.

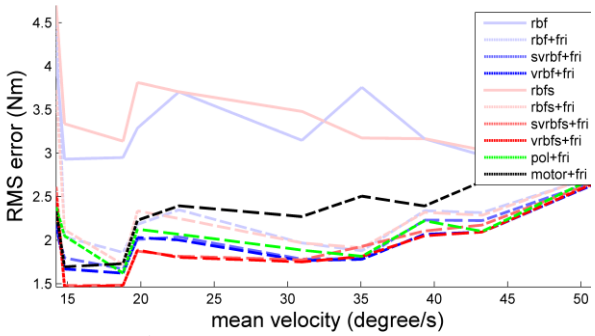

Fig. 5. RMS error of torque prediction in experiments, evaluated on the remaining 2/3 of collected data. The lines are the linear interpolations of the experimental results.

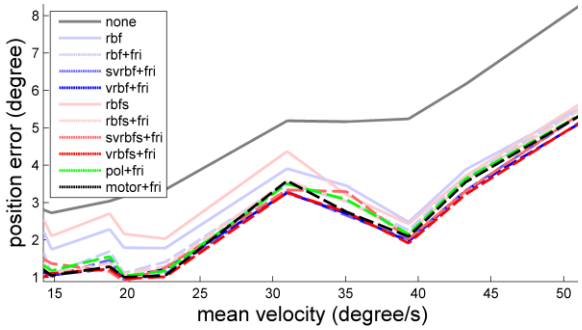

Fig. 6. RMS error of position tracking in experiments, evaluated on the remaining 2/3 of collected data. The lines are the linear interpolations of the experimental results.

Given that identically independently distributed (i.i.d.) assumption requires that the validating data (including positions, velocities, and accelerations) are sampled according to the same probability distribution as the training data, the i.i.d. assumption is not generally satisfied due to the time-domain dependency in dynamics; the learning-based methods often fail to choose the correct parameters and tend to *overfit*, if standard cross-validation is employed.

We therefore adopted blocked cross-validation [38], commonly used in time-series prediction. Blocking the training data in time domain into equal-sized groups, the optimal parameters were determined by performing the conventional cross-validation in accordance with the groups, because i.i.d. assumption is more likely to hold in terms of groups if the block is large enough.

We compare torque error in prediction and position tracking error in pre-computed torque control (i.e. feedforward dynamics compensation [39]) with the learned models to track the full trajectory used in generating the training data. We choose pre-computed torque control to limit the controller's effect on the outcome. In order to provide an unbiased estimation of the performance, we used only 500 samples from the first 1/3 of the collected trajectory in time domain in training with 5-fold 3-second blocked cross-validation, and used the remaining 2/3 as the validation set to evaluate the performance index. Therefore, the score reflects more faithfully and unbiasedly how a model may perform in applications.

Fig. 5 shows the RMS error of prediction of inverse dynamics with respect to the average norm of velocity of the trajectory. The performance is greatly improved by introducing *fri* in the semi-parametric framework. Because the friction in the NTU robot arm is large, using a single kernel *rbf* or *rbfs* standalone results in unsatisfactory results, while the simple independent joint model *motor+fri* fits well. However, as the speed increases, *motor+fri* fails, because the magnitude of the Coriolis/centrifugal forces increases. Compared to the simulation results, Fig. 5 is consistent with Fig. 3, showing that the Lagrangian kernels are the most competent among all the candidates. In particular, *svrbfs* and *vrbfs* are the optimal choices, because of the consideration of rotary joints in designing $\mathcal{H}_L$.

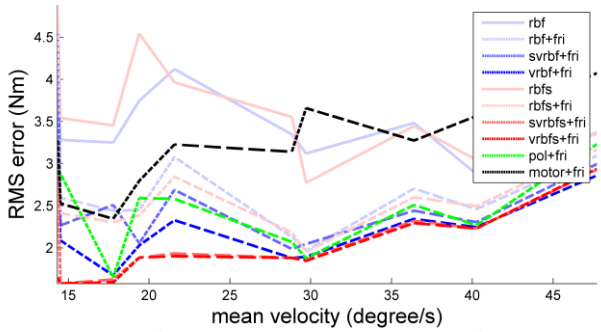

Fig. 7. RMS error of torque prediction in experiments of the weighted robot, evaluated on the remaining 2/3 of collected data. The lines are the linear interpolations of the experimental results.

Fig. 6 shows the RMS error of tracking with pre-computed torque control, where the estimated torque required for the reference trajectory is computed offline by the model as the feedforward compensation. The results in Fig. 6 provide a rather qualitatively description, because the factors, such as the torque limits of actuator or accumulated errors, affect the tracking performance. But still the results can be discriminated into three groups: without the feedforward, with the feedforward but without the friction model, and with both the feedforward and the friction model. This verifies the effectiveness of the feedforward term and the necessity of the friction model.

Finally, similar to Fig. 5, Fig. 7 shows the prediction result of the NTU robot arm with additional loads of 2.5 kg and 1 kg on the first and the second link, respectively, where the links are ordered increasingly from the base to the end-effector. In this configuration, the relative size of friction in τ_a effectively decreases. Therefore, the discrepancy between the Lagrangian kernels and the traditional approach becomes more obvious. Especially, the accuracy of *motor+fri* deteriorates much faster than that in Fig. 5 as the speed increases.

V. Discussions

In the simulations and experiments, two main factors dominate the performance: the kernel and the parametric model. First, a kernel's quality depends on both the structure of function space and the modeled correlations. Comparing *rbf* and *srbf* (or *rbfs* and s*rbfs*), though both are scalar kernels modeling each joint independently, we see that *srbf* outperforms *rbf* due to the structure inherited from Euler-Lagrange equation. In

addition, the structure results from the choice of $\mathcal{H}_L$ as well. The performances of *svrbf* and *vrbf* are far different from those of *svrbfs* and *vrbfs*, though the only difference is k_L (similar is the case of *rbf* and *rbfs*). Therefore, better models are possible by better parameterizing $\mathcal{H}_L$. On the other hand, the modeled correlation affects the performance by incorporating more information. For kernels derived from the same $\mathcal{H}_L$, we can observe the vector-valued kernels (22) (*vrbf* and *vrbfs*) are better than their scalar versions (35) (*srbf* and *srbfs*), because the vector-valued Lagrangian kernels effectively learn the system's Lagrangian with *MN* observations. Second, for systems with large friction, additional parametric models can boost the learning, as these bases are small in dimensionality compared with the models for dynamics. Similarly, further adopting the parametric rigid-body models [32, 33] as the parametric basis (e.g. [35]) in the semi-parametric framework would improve the performance; in this setting, the Lagrangian kernels correct the error due to the parameterized Lagrangian. In summary, we can analyze the learning by separating a system into two parts: the Lagrangian system and the unmodeled dynamics. Lagrangian kernels improve the learning of Lagrangian systems; additional bases or kernels are necessary if the unmodeled dynamics are large.

The results show that the Lagrangian kernels are competitive with the analytic and the learning approaches. Although we did not implement the rigid-body model in the experiments because our scenario concerns the modeling with only inputs and outputs, this argument can be concluded by comparing the performance of *motor+fri* and that of the Lagrangian kernels. Because the independent joint model approximates robot dynamics well at low speed for robots with large gear ratio (e.g. the NTU robot arm) we can posit that the performance of the rigid body models from these experiments: the Lagrangian kernels using only inputs and outputs performed nearly as good or even superior.

In addition to inverse dynamics, the model by the Lagrangian kernels can derive forward dynamics as well. As a result of Euler-Lagrange equation, by (23), the models in (32) and (33) are linear in the generalized acceleration, which can therefore be recovered by matrix inverse. This form also motivates us to consider additional constraints in learning in future works, such imposing a positive resultant inertia.

The Lagrangian kernels, however, face a tradeoff between learning rate and computational complexity in solving for the unknowns. For the dense vector-valued kernel (22), we can expect the learning rate to be faster than that of the diagonal kernels (i.e. scalar kernels), because it considers the dependency of coordinates. As shown in (B.5), if adopted, (42) effectively identifies the Lagrangian of the system by the projections on $\{T_n k_{x_i} \mid n \in \mathbb{N}_N, i \in \mathbb{N}_M\}$. Compared with independently modeling *N* outputs with the same observations, the learning with the kernel (42) has *N* times more observations and takes place in a smaller hypothesis space, because the Lagrangian consists of $2N$ variables, instead of $3N$ variables in dynamics. That is, it models $\mathcal{L}$ with *MN* samples. Therefore, although the result also depends on the distribution μ_x and the exact information provided by the coupling in K_T (which is related to the definition of k_L), we can posit that the transfer learning between different coordinates increases the convergence rate. However, the increased convergence rate is at the expense of computational complexity. Because kernel methods generally have a computational complexity $O(M^c)$ for $1 < c \le 3$, with the dense vector-valued kernels, the complexity becomes $O(N^c M^c)$. By contrast, the scalar Lagrangian kernels, e.g. (30) and (35), have the computational complexity $O(NM^c)$, because it can be decomposed into *N* independent scalar regressions. Therefore, the kernels (30) and (35) may be more favorable for large *N* from a computational viewpoint, though they model the dynamics $\{\tau_n \mid n \in \mathbb{N}_N\}$ independently and therefore may not be integrable to give a Lagrangian. Finally, we remark that though an effective $N \times MN$ kernel matrix was used in the experiments, this matrix can be further subsampled in column by exploring the inner product property RKHS [40-42]. Especially, if many samples are correlated, such unsupervised low-rank sparsification can often preserve most accuracy and drastically improve computational speed both in training and evaluation.

The Lagrangian kernels motivate kernel design from a new perspective. For identifying robot dynamics, kernels other than (38) and (39) can be adopted as k_L, for example, the finite-dimensional kernel used to model the Lagrangian of *pol* [34], which includes assumption on rigid-body robots and therefore can achieve a learning rate for general holonomic robots even faster than *vrbfs*. For Lagrangian systems, it converges to dynamics in a rate faster than general schemes, given that a set of inputs and outputs which contributes to the system's energy can be measured. More generally, it helps identify system by indirectly learning the mapping between inputs and outputs in a higher, abstracter level, or a smaller space which shares the common knowledge of all coordinates. Given with a linear map (e.g. differential operator from domain knowledge or mappings learned by machine learning) that links the two spaces, new kernels can be designed to consider such correlations or identify abstracter notions of a system, such as identifying the energy through learning gradient fields.

VI. Conclusion

We reformulate Lagrangian systems in RKHS. By treating Euler-Lagrange method as a bounded linear operator in the RKHS where the Lagrangian lies, we derive a family of Lagrangian kernels which capture inverse dynamics as the Lagrangian's projection. In application to identifying robot inverse dynamics, we incorporated the Lagrangian kernels in a semi-parametric framework; the results show that the Lagrangian kernels are competitive despite using only inputs and outputs. In essence, the Lagrangian kernels mean modelling systems by indirectly learning an abstracter notion.

APPENDIX A

Proof of Proposition 4: Equation (14) implies $\mathcal{L}(q,\dot{q})$ is second-order differentiable. Therefore, choosing a second-order differentiable $k_{L,x}$, we have

$$\tau_n(q,\dot{q},\ddot{q}) = (\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n})\left\langle \mathcal{L}, k_{L,x}\right\rangle_{\mathcal{H}_L} \tag{A.1}$$

by the reproducing property (4). Next, we show there exists $T_n k_{L,x} \in \mathcal{H}_L$ satisfying (17). Consider a function $T_n k_{L,x}(u)$ defined as

$$T_n k_{L,x}(u) \coloneqq (\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n})_x k_{L,u}(x) \tag{A.2}$$

in which the subscript denotes the respective variable the derivative takes. Because

$$k_{L,u}(x) = k_L(x,u) = \left\langle k_{L,u}, k_{L,x}\right\rangle_{\mathcal{H}_L} = \sum_i \phi_i(x)\phi_i(u) \tag{A.3}$$

for some mappings $\phi_i : X \to \mathbb{R}$, substituting (A.3) into (A.2), we have $T_n k_{L,x}(u) = \sum_i c_i \phi_i(u)$, in which $c_i = (\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n})_x \phi_i(x)$.

Therefore, $T_n k_{L,x}(u) = \left\langle T_n k_{L,x}, k_{L,u}\right\rangle_{\mathcal{H}_L}$, i.e. $T_n k_{L,x} \in \mathcal{H}_L$. Finally, by representer theorem, for $f \in \mathcal{H}_L$, let $f(x) = \int_X \alpha_u k_{L,u}(x)du$ for some α_u; we have

$$\begin{aligned}\left\langle f, T_n k_{L,x}\right\rangle_{\mathcal{H}_L} &= \left\langle T_n k_{L,x}, f\right\rangle_{\mathcal{H}_L} = \int \left\langle T_n k_{L,x}, \alpha_u k_u\right\rangle_{\mathcal{H}_L} du \\ &= (\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n})_x \int \alpha_u k_{L,u}(x)du = (\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} - \frac{\partial}{\partial q_n})_x f(x)\end{aligned} \tag{A.4}$$

Therefore, by the choice of $k_{L,x}, T_n \in L(\mathcal{H}_L)$ satisfying (16) can be defined by (17). □

Proof of Proposition 5: By construction $\mathcal{H}_{T,n}, \mathcal{H}_T \subseteq \mathcal{H}_L$. For the first part, because the equations of (16) are linear in $\mathcal{L} \in \mathcal{H}_L$, its solution admits the representation $\mathcal{L} = \mathcal{L}_p + \mathcal{L}_h : \mathcal{L}_p$ is the minimum norm solution such that (16) holds, for all $n \in \mathbb{N}_N$; $\mathcal{L}_h$ is the null space solution, which has no effect on the evaluation of $\tau_n(q,\dot{q},\ddot{q})$. And by construction of (20), $\mathcal{L}_p \in \mathcal{H}_T$ and $\mathcal{L}_h \in \mathcal{H}_L \setminus \mathcal{H}_T$. For the second part, endow $\mathcal{H}_{T,n}$ with $\left\langle \cdot,\cdot\right\rangle_{\mathcal{H}_{T,n}} \coloneqq \left\langle \cdot,\cdot\right\rangle_{\mathcal{H}_L}$; then $\mathcal{H}_{T,n}$ is a RKHS with $k_{Tn,x} : \mathcal{X} \to \mathcal{H}_{T,n}$ by Definition 2 and (21) follows from (16) by property of RKHS. For the last part, because the projection of $\mathcal{L}_p$ on $\mathcal{H}_{T,n}$ satisfies (16) and the uniqueness of τ_n, τ_n equals to the projection of $\mathcal{L}$ on the $\mathcal{H}_{T,n}$. □

APPENDIX B

The convergence of the regularized framework can be proved by the compactness of the hypothesis space [43]: there is $R(\gamma)$, a non-increasing function of γ, such that the solution of (42) equals to the solution of

$$\min_{f \in \mathcal{H}_{dyn} \oplus \mathcal{H}_{fri}} \frac{1}{M}\sum_{i \in \mathbb{N}_m} \left\| f(x_i) - y_i\right\|_y^2 \ s.t.\ \left\| f\right\|_{\mathcal{H}_{dyn} \oplus \mathcal{H}_{fri}} \le R \text{ ; (B.1)}$$

if $\gamma > 0$, then $R < \infty$. That is, (B.1) minimizes the cost function in a compact convex hypothesis space with a unique solution.

Consider a hypothesis $g = g_{dyn} + g_{fri} \in \mathcal{H}_{dyn} \oplus \mathcal{H}_{fri}$, where

$$g_{fri} \coloneqq \arg\min_{f \in \mathcal{H}_{fri}} \int_X \left\| f(x) + \tau_f\right\|_y^2 d\mu_x \tag{B.2}$$

$$g_{dyn} \coloneqq \arg\min_{f \in \mathcal{H}_{dyn}} \int_X \left\| f(x) - \tau\right\|_y^2 d\mu_x \ . \tag{B.3}$$

First, we prove $\left\| g\right\|_{\mathcal{H}_{dyn} \oplus \mathcal{H}_{fri}} \le \infty$. Because

$$\left\| g\right\|_{\mathcal{H}_{dyn} \oplus \mathcal{H}_{fri}}^2 = (1-\rho)\left\| g_{dyn}\right\|_{\mathcal{H}_{dyn}}^2 + \rho\left\| g_{fri}\right\|_{\mathcal{H}_{fri}}^2, \tag{B.4}$$

it is sufficient to bound $\left\| g_{dyn}\right\|_{\mathcal{H}_{dyn}}$ and $\left\| g_{fri}\right\|_{\mathcal{H}_{fri}}$. For (22), by (32) we have

$$\left\| g_{dyn}\right\|_{\mathcal{H}_{dyn}} = \left\| \int_{\mathcal{X}} K_{T,u}\alpha(u)du\right\|_{\mathcal{H}_{dyn}} = \left\| \sum_{n \in \mathbb{N}_N} \int_{\mathcal{X}} \alpha_n(u) T_n k_{L,u} du\right\|_{\mathcal{H}_L} = \left\| \mathcal{L}_p\right\|_{\mathcal{H}_L} < \infty \tag{B.5}$$

For (30), because (42) can be decoupled into N independent scalar regressions in the space $\mathcal{H}_{T,n} \subseteq \mathcal{H}_L$ learning the projection of $\mathcal{L}$ on $\mathcal{H}_{T,n}$ as in Proposition 5**,**

$$\left\| g_{dyn}\right\|_{\mathcal{H}_{dyn}} \le N \max_{n \in \mathbb{N}_N} \left\| \tau_n\right\|_{\mathcal{H}_{T,n}} = N\left\| \mathcal{L}_p\right\|_{\mathcal{H}_L} < \infty \text{ ;} \tag{B.6}$$

for(35), because the spaces $\mathcal{H}_{T,n}$ are derived from $\mathcal{H}_{L,n}$, similar results can be obtained. On the other hand, $\left\| g_{fri}\right\|_{\mathcal{H}_{fri}} < \infty$, because $\left\| \tau_f\right\|_y < \infty$, $\dim(\mathcal{H}_{fri}) < \infty$, and the covering ball in finite-dimensional space is finite [43]. Thus, there is $0 < R^* < \infty$ such that $\left\| g\right\|_{\mathcal{H}_{dyn} \oplus \mathcal{H}_{fri}} \le R^*$.

Let $\gamma^* = R^{-1}(R^*)$. Then for $0 < \gamma < \gamma^*$, the error function of $f_{\mu,\gamma} = f_{\mu,\gamma,dyn} + f_{\mu,\gamma,fri}$ can be bounded by

$$\begin{aligned}&\int_X \left\| f_{\mu,\gamma} - \tau_a\right\|_y^2 d\mu_x \\ &= \int_X \left\| f_{\mu,\gamma,dyn} - \tau\right\|_y^2 + \left\| f_{\mu,\gamma,fri} + \tau_f\right\|_y^2 + 2\left\langle f_{\mu,\gamma,dyn} - \tau, f_{\mu,\gamma,fri} + \tau_f\right\rangle_y d\mu_x \\ &\le \int_X \left\| g_{dyn} - \tau\right\|_y^2 + \left\| g_{fri} + \tau_f\right\|_y^2 + 2\left\langle g_{\mu,dyn} - \tau, g_{\mu,fri} + \tau_f\right\rangle_y d\mu_x \\ &= \int_X \left\| g_{\mu,fri} + \tau_f\right\|_y^2 d\mu_x\end{aligned} \tag{B.7}$$

since $f_{\mu,\gamma}$ is the minimizer of (B.1) for $R(\gamma^*)$ and $g_{\mu,dyn}$ converges to τ pointwisely. Thus, by the law of large numbers and the monotonicity of γ, the rest of the theorem follows. □

REFERENCES

[1] Z. Li, Y. Zhang, and Y. Yang, "Support vector machine optimal control for mobile wheeled inverted pendulums with unmodelled dynamics " *Neurocomputing*, vol. 73, pp. 2773-2782, Aug. 2010.

[2] H. A. Kingravi, G. Chowdhary, P. A. Vela, and E. N. Johnson, "Reproducing kernel Hilbert space approach for the online update of

radial bases in neuro-adaptive control," *IEEE Trans. Neural Netw. and Learn. Syst.*, vol. 23, pp. 1130-1141, Jul. 2012.
[3] F. Abdollahi, H. A. Talebi, and R. V. Patel, "Stable identification of nonlinear systems using neural networks: theory and experiments," *IEEE Trans. Mechatron.*, vol. 11, pp. 488-495, Aug. 2006.
[4] A. Yazdizadeh, K. Khorasani, and R. V. Patel, "Identification of a two-link flexible manipulator using adaptive time delay neural networks," *IEEE Tran. Syst., Man, Cybern. B, Cybern.* , vol. 30, pp. 165-172, Feb 2000.
[5] D. Kukolj and E. Levi, "Identification of complex systems based on neural and Takagi-Sugeno fuzzy model," *IEEE Trans. Syst. Man Cybern. B, Cybern.*, vol. 34, pp. 272-282, Feb. 2004.
[6] C. E. Rasmussen and C. K. I. Williams, *Gaussian Processes for Machine Learning*. Cambridge, MA: MIT Press, 2006.
[7] J. Zhu, S. C. H. Hoi, and M. R.-T. Lyu, "Robust Regularized Kernel Regression," *IEEE Trans. Syst. Man Cybern. B, Cybern.*, vol. 36, pp. 1639-1644, Dec. 2008.
[8] G.-B. Hunag, H. Zhou, X. Ding, and R. Zhang, "Extreme learning machine for regression and multiclass classification," *IEEE Trans. Syst. Man Cybern. B, Cybern.*, vol. 42, pp. 513-529, Apr. 2012.
[9] B. Schoelkopf and A. J. Smola, *Learning with Kernels*. Cambridge, MA: The MIT Press, 2001.
[10] A. Caponnetto, C. A. Micchelli, M. Pontil, and Y. Ying, "Universal multi-task kernels," *J. Mach. Learning Research*, vol. 9, pp. 1615-1646 Jul 2008.
[11] E. Vazquez and E. Walter, "Multi-output support vector regression," in *13th IFAC Symp. System Identification*, De Doelen, Rotterdam, 2003.
[12] S. Xu, X. An, X. Qiao, L. Zhu, and L. Li, "Multi-output least-squares support vector regression machines," *Pattern Recognition Lett.*, vol. 34, pp. 1078-1084, Jul. 2013.
[13] D. Tuia, J. Verrelst, L. Alonso, F. Perez-Cruz, and G. Camps-Valls, "Multioutput support vector regression for remote sensing biophysical parameter estimation," *IEEE Geosci. Remote Sens. Lett.*, vol. 8, pp. 804-808, Jul. 2011.
[14] M. Sanchez-Fernandez, M. de-Prado-Cumplido, J. Arenas-Garcia, and F. Perez-Cruz, "SVM multiregression for nonlinear channel estimation in multiple-input multiple-output systems," *IEEE Trans. Signal Process.*, vol. 52, pp. 2298 - 2307, Aug. 2004.
[15] S. Vijayakumar and S. Schaal, "Locally weighted projection regression: an O(n) algorithm for incremental real time Learning in high dimensional space," in *17th Int. Conf. Machine Learning* Stanford, CA, USA, 2000, pp. 1079-1086.
[16] G. Pedrick, *Theory of Reproducing Kernels for Hilbert Spaces of Vector Valued Functions*. Kansas: Univ. Kansas Dep. Math., 1957.
[17] J. Burbea and P. R. Masani, *Banach and Hilbert Spaces of Vector-Valued Functions*. London, UK: Pitman Publishing, 1983.
[18] B. Schoelkopf, R. Herbrich, and A. J. Smolar, "A generalized representer theorem," in *Annu. Conf. Computational Learning Theory*, Amsterdam, the Netherlands, 2001, pp. 416-426.
[19] T. Evgeniou, C. A. Micchelli, and M. Pontil, "Learning Multiple Tasks with Kernel Methods," *J. Mach. Learning Research*, vol. 6, pp. 615-637, Apr. 2005.
[20] C. Carmeli, E. D. Vito, and A. Toigo, "Vector valued reproducing kernel Hilbert spaces of integrable functions and Merer theorem," *Anal. and Applicat.*, vol. 4, pp. 377-408, Oct. 2006.
[21] C. Carmeli, E. De Vito, A. Toigo, and V. Umanità, "Vector valued reproducing kernel Hilbert spaces and universaltiy," *Anal. and Applicat.*, vol. 8, pp. 19-61, Jan. 2010.
[22] A. Caponnetto and E. D. Vito, "Optimal rates for the regularized least-squares algorithm," *Found. Computational Math.*, vol. 7, pp. 331-368, Jul. 2007.
[23] M. Reisert and H. Burkhardt, "Learning equivariant functions with matrix valued kernels," *J. Mach. Learning Research*, vol. 8, pp. 385-408, Mar. 2007.
[24] C. A. Micchelli and M. Pontil, "On learning vector-valued functions," *Neural Computation*, vol. 17, pp. 177-204, Jan 2005.
[25] Z. Guo and J. Wang, "Cross-domain object recognition via input-output kernel analysis," *IEEE Trans. Image Process.*, vol. 22, pp. 3108-3119, Aug. 2013.
[26] Y. Luo, D. Tao, C. Xu, C. Xu, H. Liu, and Y. Wen, "Multiview vector-valued manifold regularization for multilabel image classification," *IEEE Trans. Neural Netw. and Learn. Syst.* , vol. 24, pp. 709-722, May 2013.
[27] Z. Lu, J. Sun, and K. Butts, "Mutiscale asymmetric orthogonal wavelet kernel for linear programming support vector learning and nonlinear dynamic systems identification," *IEEE Trans. Cybern.*, vol. 44, pp. 712-724, May 2014.
[28] S. Chiaverini, B. Siciliano, and L. Villani, "A survey of robot interaction control schemes with experimental comparison," *IEEE Trans. Mechatron.*, vol. 4, pp. 273-285, Sep 1999.
[29] C. Ott, *Cartesian Impedance Control of Redundant and Flexible-Joint Robots* vol. 49. New York: Springer, 2008.
[30] T.-H. Huang, C.-A. Cheng, and H.-P. Huang, "Self-learning assistive exoskeleton with sliding mode admittance control," in *IEEE/RSJ Int. Conf. Intelligent Robots and Systems*, Tokyo, Japan, 2013, pp. 698-703.
[31] H. Goldstein, C. P. P. Jr., and J. L. Safko, *Classical Mechanics*, 3rd ed. Boston: Addison-Wesley, 2001.
[32] M. Gautier and W. Khalil, "On the identification of the inertial parameters of robots," in *27th IEEE Conf. Decision and Control*, Austin, TX, 1988, pp. 2264-2269.
[33] C. G. Atkeson, C. H. An, and J. M. Hollerbach, "Estimation of inertial parameters of manipulator loads and links," *Int. J. Robotics Research*, vol. 5, pp. 101-119, Sep. 1986.
[34] C.-A. Cheng, H.-P. Huang, H.-K. Hsu, W.-Z. Lai, and C.-C. Chung, "Learning the Inverse Dynamics of Robotic Manipulators in Structured Reproducing Kernel Hilbert Space," *IEEE Trans. Cybern.*, 2014, submitted for publication.
[35] N.-T. Duy and J. Peters, "Using model knowledge for learning inverse dynamics," in *IEEE Int. Conf. Robotics and Automation*, Anchorage, Alaska, USA, 2010, pp. 2677-2682
[36] B. Siciliano and O. Khatib, *Springer Handbook of Robotics*. New York: Springer, 2008.
[37] G. Liu and Z. Li, "A unified geometric approach to modeling and control of constrained mechanical systems," *IEEE Trans. Robot. Autom.*, vol. 4, pp. 574-587, Aug. 2002.
[38] C. Bergmeir and J. M. Benitez, "On the use of cross-validation for time series predictor evaluation," *Inform. Sciences*, vol. 191, pp. 192-213, May 2012.
[39] P. K. Khosla and T. Kanade, "Experimental evaluation of nonlinear feedback and feedforward control schemes for manipulators," *Int. J. Robotics Research*, vol. 7, pp. 18-28, Feb. 1988.
[40] P. B. Nair, A. Choudhury, and A. J. Keane, "Some Greedy Learning Algorithms for Sparse Regression and Classification with Mercer Kernels," *J. Mach. Learning Research*, vol. 3, pp. 781-801, Dec. 2002.
[41] Y. Engel, S. Mannor, and R. Meir, "The kernel recursive least-squares algorithm," *IEEE Trans. Signal Process.*, vol. 52, pp. 2275-2285, Aug. 2004.
[42] D. Nguyen-Tuong and J. Peters, "Incremental online sparsification for model learning in real-time robot control," *Neuralcomputing*, vol. 74, pp. 1859-1867, 2011.
[43] F. Cucker and D. X. Zhou, *Learning Theory - An Approximation Theory Viewpoint*. New York: Cambridge University Press, 2007.

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